HORIZONTAL POSITIONAL ACCURACY FROM DIRECTIONAL ANTENNA

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ABSTRACT

To monitor the position of athletes during sporting contests it is proposed to equip the athletes with small radio transmitters whose signal is received by several directional radio antennae located at known locations. This paper provides a method of determining the athletes horizontal position as well as the precision of position given the precision of the directional measurements. In addition an analysis of precision of position fixing for different antennae locations is provided

INTRODUCTION

Suppose an athlete, equipped with a small radio transmitter, is located within the range of several directional radio antennae. The antennae can detect the transmitted signal and determine the bearing of the line between the antennae and the athlete. Figure 1 shows an athlete at P and four directional antennae at positions A, B, C and D whose East and North coordinates E,N are known. The bearings (clockwise angles from North) ϕ_A, ϕ_B, ϕ_C and ϕ_D of the lines from the antennae to athlete are shown.

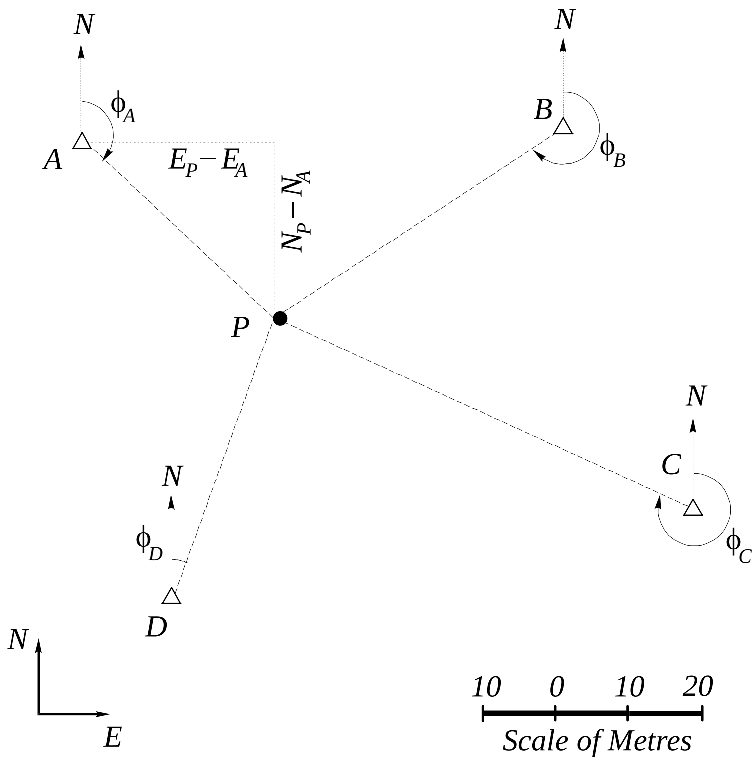

Figure 1. Athlete at P and bearings from antennae locations A, B, C and D

The *E,N* coordinates of the athlete at *P* can be determined from any two bearings from the directional antennae locations. A formula for computing the coordinates of *P* from two bearings can be obtained from coordinate geometry and is set out in a following section. If three or more bearings from antennae to athlete are observed then the determination of the athletes position is more complicated. This paper details a method, using the *least squares* technique, that provides the "best estimate" of the athletes position using all the observed bearings in a series of simultaneous linear equations. In addition, it will also be shown that a least squares solution provides estimates of the precision of the computed position and that this precision estimate is a function of (i) the number of antennae, (ii) the location of the antennae, (iii) the precision of the measured bearings and (iv) the position of the athlete.

POSITION OF ATHLETE FROM TWO OBSERVED BEARINGS

From Figure 1, using the bearings ϕ_A, ϕ_B to the athlete at *P* from antennae *A* and *B* at locations E_A, N_A and E_B, N_B the following two equations can be obtained

$$\begin{aligned} \tan\phi_A &= \frac{E_P - E_A}{N_P - N_A} \\ \tan\phi_B &= \frac{E_P - E_B}{N_P - N_B} \end{aligned} \qquad (1)$$

Expanding these equations and re-arranging gives

$$\begin{aligned} E_P &= \tan\phi_A N_P - \tan\phi_A N_A + E_A \\ E_P &= \tan\phi_B N_P - \tan\phi_B N_A + E_B \end{aligned} \qquad (2)$$

Equating equations (2) gives a solution for N_P

$$N_P = \frac{\tan\phi_A N_A - \tan\phi_B N_B + E_B - E_A}{\tan\phi_A - \tan\phi_B} \qquad (3)$$

Having obtained a solution for N_P from (3) then E_P can be obtained from either of equations (2).

It should be noted that if *P* lies on the line between *A* and *B* then its position is indeterminate.

LEAST SQUARES POSITION OF ATHLETE FROM THREE OR MORE OBSERVED BEARINGS

Using equations (2) and (3) the coordinates of an athlete at *P* can be determined from pairs of observed bearings, ie from four bearings there would be six possible solutions for the coordinates of *P*.

For *n* bearings taken in pairs $(m = 2)$, there will be $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ possible solutions.

A crude solution for the athlete's position is to take an average of all possible solutions, but this method will not give the "best estimate" nor will it provide any information on the precision of the computed solution.

A better approach is to use the principle of *least squares*. A least squares solution depends upon the formation of a set of *observation equations* and their solution.

$$\phi_k + v_k = \tan^{-1}\left(\frac{E_P - E_k}{N_P - N_k}\right) = f\left(E_k, N_k, E_P, N_P\right) \qquad (4)$$

where ϕ_k are observed bearings from the antennae locations *k* to the athlete at *P*,

v_k are residuals (small corrections) associated with observed bearings,

E_k, N_k are east and north coordinates of the antennae locations and

E_P, N_P are east and north coordinates of the athlete.

Equation (4) expresses the fact that every observed bearing has a small, unknown correction (residual), which if added to the observation would yield the "true" bearing. The true bearing is a function of the known coordinates of the antennae and the unknown coordinates of the athlete. An observation equation can be written for each observed bearing and this set of equations can be solved by enforcing the least squares principle. This principle states that the best estimate of the coordinates is that which makes the sum of the squares of the residuals, multiplied by coefficients expressing the precision of the observations, a minimum. This technique first employed by the German mathematician C.F. Gauss in 1795, is used extensively in surveying applications.

The normal techniques of solution of systems of equations require that the sets of observation equations must be *linear,* ie, "unknowns" linearly related to measurements. This is not the case in this problem where the observed bearings ϕ_k (the measurements) are non-linear functions of the coordinate differences. In equation (4), the measurements ϕ_k are non-linear functions of the unknowns E_P, N_P and any system of equations based on (4) would be *non-linear* and could not be solved by normal means. Equation (4) can be linearized and the most common method is to use *Taylor's* theorem to represent the function as a power series with zero order terms, 1st order terms, and higher order terms. By choosing suitable approximations, the higher-order terms can be neglected, yielding a linear approximation to the function. This linear approximation of the mathematical model can be used to form a set of linear equations, which can be solved by normal means.

Taylor's theorem can be used to expand a non-linear function into a linear series. Consider a function of a single variable *x*, Taylor's theorem gives a convergent power series for $f\left(x\right)$ about the point $x = a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \quad (5)$$

where R_n is the remainder after *n* terms and $\lim_{n\to\infty} R_n = 0$ for $f\left(x\right)$ about $x = a$

$f'(a),\ f''(a), \cdots$ etc are derivatives of the function $f\left(x\right)$ evaluated at $x = a$.

Suppose that the athlete's position E_P, N_P is given by $E_P = E'_P + \Delta E_P$ and $N_P = N'_P + \Delta N_P$ where E'_P, N'_P are approximate coordinates and $\Delta E_P, \Delta N_P$ are small corrections. Taylor's theorem can be used to linearize equation (4) to give

$$\phi_k + v_k = \phi'_k + \left(E_P - E'_P\right)\frac{\partial \phi_k}{\partial E_P} + \left(N_P - N'_P\right)\frac{\partial \phi_k}{\partial N_P} + \left(E_P - E'_P\right)^2\frac{\partial^2 \phi_k}{\partial E_P^2} + \left(N_P - N'_P\right)^2\frac{\partial^2 \phi_k}{\partial N_P^2} + \cdots \qquad (6)$$

ϕ_k' is the bearing to the athlete computed using the approximate coordinates of the athlete's position and $\dfrac{\partial \phi_k}{\partial E_P}, \dfrac{\partial \phi_k}{\partial N_P}$ are partial derivatives of the function evaluated using the approximate coordinates.

In equation (6) the terms in parentheses are corrections $\Delta E_P = E_P - E_P'$ and $\Delta N_P = N_P - N_P'$. If the approximate coordinates are close to the actual coordinates then the corrections will be small and powers of these corrections will be exceedingly small and may be neglected in a linear approximation of the form

$$\phi_k + v_k = \phi_k' + \Delta E_P \frac{\partial \phi_k}{\partial E_P} + \Delta N_P \frac{\partial \phi_k}{\partial N_P} \tag{7}$$

The partial derivatives in equation (7) are evaluated in the following manner.

Using the relationships: $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ and $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{\partial \phi_k}{\partial E_P} = \frac{1}{1+\left(\dfrac{E_P - E_k}{N_P - N_k}\right)^2}\frac{\partial}{\partial E_P}\left(\frac{E_P - E_k}{N_P - N_k}\right) = \frac{\left(N_P - N_k\right)^2}{\left(N_P - N_k\right)^2 + \left(E_P - E_k\right)^2}\frac{N_P - N_k}{\left(N_P - N_k\right)^2}$$

giving

$$\frac{\partial \phi_k}{\partial E_P} = \frac{N_P - N_k}{\left(N_P - N_k\right)^2 + \left(E_P - E_k\right)^2} = \frac{N_P - N_k}{s_k^2} = \frac{\cos\phi_k}{s_k} = b_k \tag{8a}$$

Similarly

$$\frac{\partial \phi_k}{\partial N_P} = \frac{-\left(E_P - E_k\right)}{\left(N_P - N_k\right)^2 + \left(E_P - E_k\right)^2} = \frac{-\left(E_P - E_k\right)}{s_k^2} = \frac{-\sin\phi_k}{s_k} = a_k \tag{8b}$$

Substituting equations (8) into (7) and re-arranging gives a linearized observation equation

$$v_k - b_k \Delta E_P - a_k \Delta N_P = \phi_k' - \phi_k \tag{9}$$

For *n* observed bearings, the equations in the *u* unknowns can be written in matrix form as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} -b_1 & -a_1 \\ -b_2 & -a_2 \\ -b_3 & -a_3 \\ \vdots & \vdots \\ -b_n & -a_n \end{bmatrix}\begin{bmatrix} \Delta E_P \\ \Delta N_P \end{bmatrix} = \begin{bmatrix} \phi_1' - \phi_1 \\ \phi_2' - \phi_2 \\ \phi_3' - \phi_3 \\ \vdots \\ \phi_n' - \phi_n \end{bmatrix} \tag{10a}$$

or

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{10b}$$

where
- n is the number of equations,
- u is the number of unknowns (the corrections to the approximate coordinates),
- $\mathbf{v}$ is an $(n,1)$ vector of residuals,
- $\mathbf{B}$ is an (n,u) coefficient matrix containing the coefficients *a* and *b*,
- $\mathbf{x}$ is a $(u,1)$ vector of corrections to approximate coordinates and

f is an $(n,1)$ vector of numeric terms (computed bearing minus observed bearing).

Each observed bearing has an associated estimate of precision. Precisions are usually expressed as variances σ_ϕ^2 or standard deviations σ_ϕ (the positive square root of the variance) and estimates of these quantities are denoted by s_ϕ^2 and s_ϕ. In least squares solutions the measurements are assumed to be random variables and their statistical connection, the covariance denoted by σ_{jk} and estimated by s_{jk}, must be considered. In most practical applications the measurements are treated as independent and hence their covariances are zero. For a set of *n* measurements, the variance matrix $\mathbf{\Sigma}$ (containing variances and covariances) is estimated by the cofactor matrix **Q** (containing estimates of the variances and covariances). Variance matrices and cofactor matrices are related by

$$\mathbf{\Sigma} = \sigma_0^2 \mathbf{Q}$$

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{2n} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \cdots & \sigma_{3n} \\ \vdots & & & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \sigma_{3n} & \cdots & \sigma_{nn}^2 \end{bmatrix} = \sigma_0^2 \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_2^2 & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_3^2 & \cdots & s_{3n} \\ \vdots & & & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn}^2 \end{bmatrix} \qquad (11)$$

where σ_0^2 is the variance factor. In many least squares applications, it is usual to express the precision of measurements in terms of weights where a weight *w* is defined as being inversely proportional to an estimate of variance s^2; measurements of high weight having low precision. A weight matrix **W** is defined as the inverse of the cofactor matrix **Q**

$$\mathbf{W} = \mathbf{Q}^{-1} \qquad (12)$$

The system of equations (10), *n* equations in *u* unknowns $n > u$, is solved by employing the least squares principle, ie a solution for the corrections $\Delta E_P, \Delta N_P$ is determined such that the sum of the squares of the residuals (multiplied by precision coefficients) is a minimum. This principle may be expressed mathematically as the minimisation of a function φ where

$$\varphi = \mathbf{v}^T \mathbf{W} \mathbf{v} = \left(\mathbf{f} - \mathbf{B}\mathbf{x}\right)^T \mathbf{W} \left(\mathbf{f} - \mathbf{B}\mathbf{x}\right) \qquad (13)$$

Differentiating φ with respect to the unknowns **x** and equating the derivatives to zero leads to a set of *normal equations*

$$\left(\mathbf{B}^T \mathbf{W} \mathbf{B}\right)\mathbf{x} = \mathbf{B}^T \mathbf{W} \mathbf{f} \qquad (14a)$$

or

$$\mathbf{N}\mathbf{x} = \mathbf{t} \qquad (14b)$$

The solution for the unknowns **x** (corrections $\Delta E_P, \Delta N_P$) is given by

$$\begin{aligned} \mathbf{x} &= \left(\mathbf{B}^T \mathbf{W} \mathbf{B}\right)^{-1} \mathbf{B}^T \mathbf{W} \mathbf{f} \\ &= \mathbf{N}^{-1}\mathbf{t} \end{aligned} \qquad (15)$$

The vector **x** contains corrections to approximate positions E_P', N_P' based on a linearized approximation of the observation equation (4). The smaller the corrections, the closer the linearized equation approximates the non-linear equation. Thus, the correct solution is obtained by iteration.

EXAMPLE COMPUTATION OF ATHLETE'S POSITION

From Figure 1 the coordinates of the four antennae locations and the bearings to the athlete are

Point	East	North
A	6.00	79.06
B	72.50	81.03
C	90.42	28.29
D	18.36	16.15

Point	Bearing	Estimated st.dev. s_ϕ
ϕ_A	$132^\circ\,30'$	$0^\circ 10'$
ϕ_B	$236^\circ\,20'$	$0^\circ 10'$
ϕ_C	$294^\circ\,00'$	$0^\circ 10'$
ϕ_C	$20^\circ\,40'$	$0^\circ 10'$

Using equations (2) and (3) and the bearings ϕ_A and ϕ_B the approximate position of the athlete is

$$N_P = \frac{\tan\phi_A N_A - \tan\phi_B N_B + E_B - E_A}{\tan\phi_A - \tan\phi_B} = 54.551$$

$$E_P = \tan\phi_A N_P - \tan\phi_A N_A + E_A = 32.747$$

Using the approximate position of the athlete, the computed bearings ϕ' and distances s' are

Point	Computed Bearing	Computed Distance
ϕ_A	$132^\circ\,30'00''$	36.278
ϕ_B	$236^\circ\,19'58''$	47.765
ϕ_C	$294^\circ\,28'55''$	63.371
ϕ_C	$20^\circ\,32'19''$	41.008

The set of observation equations in the form of (10) are given below. Note that the elements of the coefficient matrix **B** are in units of sec/cm, the corrections $\Delta E_P, \Delta N_P$ are in cm's and the numeric terms **f** are in seconds.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 38.4118 & 41.9192 \\ 23.9394 & -35.9402 \\ -13.4884 & -29.6224 \\ -47.1015 & 17.6467 \end{bmatrix} \begin{bmatrix} \Delta E_P \\ \Delta N_P \end{bmatrix} = \begin{bmatrix} 0'' \\ -2'' \\ 1735'' \\ -461'' \end{bmatrix} \qquad (16)$$

Using (15), the solution for **x** is

$$\mathbf{x} = \begin{bmatrix} \Delta E_P \\ \Delta N_P \end{bmatrix} = \begin{bmatrix} 0.6164 \\ -14.0767 \end{bmatrix}$$

The corrections (cm's) applied to the approximate coordinates give improved estimates $E_P = E'_P + \Delta E_P = 32.747 + 0.006 = 32.753$ and $N_P = N'_P + \Delta N_P = 54.551 - 0.141 = 54.410$.
Using these values as approximate coordinates in another iteration gives corrections less than 0.5 mm so for all practical purposes the results from the first iteration can be regards as exact.

PRECISION OF COMPUTED ATHLETE POSITION

A very useful property of a least squares solution is that estimates of the precision of the computed quantities **x** is contained in the inverse of the coefficient matrix **N** of the normal equations (14). That is

$$\mathbf{Q}_{xx} = \mathbf{N}^{-1} \tag{16a}$$

In our case of intersecting bearings defining the athletes position E_P, N_P

$$\begin{bmatrix} s_E^2 & s_{EN} \\ s_{EN} & s_N^2 \end{bmatrix} = \mathbf{N}^{-1} \tag{16b}$$

This property can be established by using the *Law of Propagation of Variances,* or the Law of Propagation of Cofactors since cofactor matrices are estimates of variance matrices. This law states that if random variables **x** and **y** are related by

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{17}$$

where **A** is a coefficient matrix and **b** is a vector of numeric terms (or constants) then the cofactor matrix $\mathbf{Q}_{yy}$ is given by

$$\mathbf{Q}_{yy} = \mathbf{A}\mathbf{Q}_{xx}\mathbf{A}^T \tag{18}$$

The sequence of equations in a least squares solution may be summarised as

$$\begin{aligned} \mathbf{f} &= \mathbf{d} - \mathbf{l} \\ \mathbf{N} &= \mathbf{B}^T\mathbf{W}\mathbf{B} \\ \mathbf{t} &= \mathbf{B}^T\mathbf{W}\mathbf{f} \\ \mathbf{x} &= \mathbf{N}^{-1}\mathbf{t} \end{aligned} \tag{19}$$

The first equation is the numeric terms **f** equal to computed bearings **d** minus observed bearings **l**. This equation can be re-cast as

$$\mathbf{f} = -(\mathbf{I})\mathbf{l} + \mathbf{d}$$

where **I** is the identity matrix and the terms in parentheses represents the coefficient matrix **A** in equation (17). Applying the Law of Propagation of Variances gives

$$\mathbf{Q}_{ff} = (-\mathbf{I})\mathbf{Q}_{ll}(-\mathbf{I})^T = \mathbf{Q}_{ll} = \mathbf{Q} \tag{20}$$

$\mathbf{Q}_{ll}$ is the cofactor matrix of the observations **l** and the subscript "*ll*" is dropped from $\mathbf{Q}_{ll}$ and $\mathbf{W}_{ll}$ in these notes.

The third equation of (19) can be written as

$$\mathbf{t} = \left(\mathbf{B}^T\mathbf{W}\right)\mathbf{f}$$

Noting that $\mathbf{W} = \mathbf{Q}^{-1}$ and $\mathbf{W}^T = \mathbf{W}$ (since **W** and **Q** are symmetric) and using (20) the propagation law gives

$$\begin{aligned}\mathbf{Q}_{tt} &= \left(\mathbf{B}^T\mathbf{W}\right)\mathbf{Q}_{ff}\left(\mathbf{B}^T\mathbf{W}\right)^T \\ &= \mathbf{B}^T\mathbf{WQW}^T\mathbf{B} \\ &= \mathbf{B}^T\mathbf{WB} \\ &= \mathbf{N}\end{aligned} \tag{21}$$

The last equation of (19) can be written as

$$\mathbf{x} = \left(\mathbf{N}^{-1}\right)\mathbf{t}$$

Noting that $\mathbf{N}$ and $\mathbf{N}^{-1}$ are symmetric and using (21), the propagation law gives the proof of (16a)

$$\begin{aligned}\mathbf{Q}_{xx} &= \left(\mathbf{N}^{-1}\right)\mathbf{Q}_{tt}\left(\mathbf{N}^{-1}\right)^T \\ &= \mathbf{N}^{-1}\mathbf{N}\mathbf{N}^{-1} \\ &= \mathbf{N}^{-1}\end{aligned}$$

In the computation example above, the inverse of the normal equation coefficient matrix is

$$\mathbf{N}^{-1} = \mathbf{Q}_{xx} = \begin{bmatrix} s_E^2 & s_{EN} \\ s_{EN} & s_N^2 \end{bmatrix} = \begin{bmatrix} 81.3530 & -6.1080 \\ -6.1080 & 85.4081 \end{bmatrix}$$

and the estimates of variance and standard deviation of position are

$$\begin{aligned} s_E^2 &= 81.3530 \text{ cm}^2 \quad \text{and} \quad s_E = 0.090 \text{ m} \\ s_N^2 &= 85.4081 \text{ cm}^2 \quad \text{and} \quad s_N = 0.092 \text{ m}\end{aligned}$$

ERROR ELLIPSES

Error ellipses are a graphical representation of the precision of the computed coordinates. They can be used to gauge the "strength" of the position fix; circular ellipses indicate a strong position fix and elongated ellipses indicate a weak position fix.

Mikhail (1976, pp.30-31) gives the equations for the lengths of the semi-axes *a* and *b* of the error ellipse as

$$a^2 = \frac{1}{2}\left(s_E^2 + s_N^2 + \sqrt{\left(s_E^2 - s_N^2\right)^2 + \left(2\,s_{EN}\right)^2} \right) \tag{22a}$$

$$b^2 = \frac{1}{2}\left(s_E^2 + s_N^2 - \sqrt{\left(s_E^2 - s_N^2\right)^2 + \left(2\,s_{EN}\right)^2} \right) \tag{22b}$$

and the angle θ between east axis and the major axis of the error ellipse as

$$\tan 2\theta = \frac{2\,s_{EN}}{s_E^2 - s_N^2} \tag{23}$$

The correct quadrant of 2θ is determined from the signs of the numerator and denominator in equation (23) and θ is measured positive anticlockwise from the east axis.

For the example above the parameters of the error ellipse are

$$a = 9.46 \text{ cm}$$
$$b = 8.79 \text{ cm}$$
$$\theta = 125.8 \text{ degrees}$$

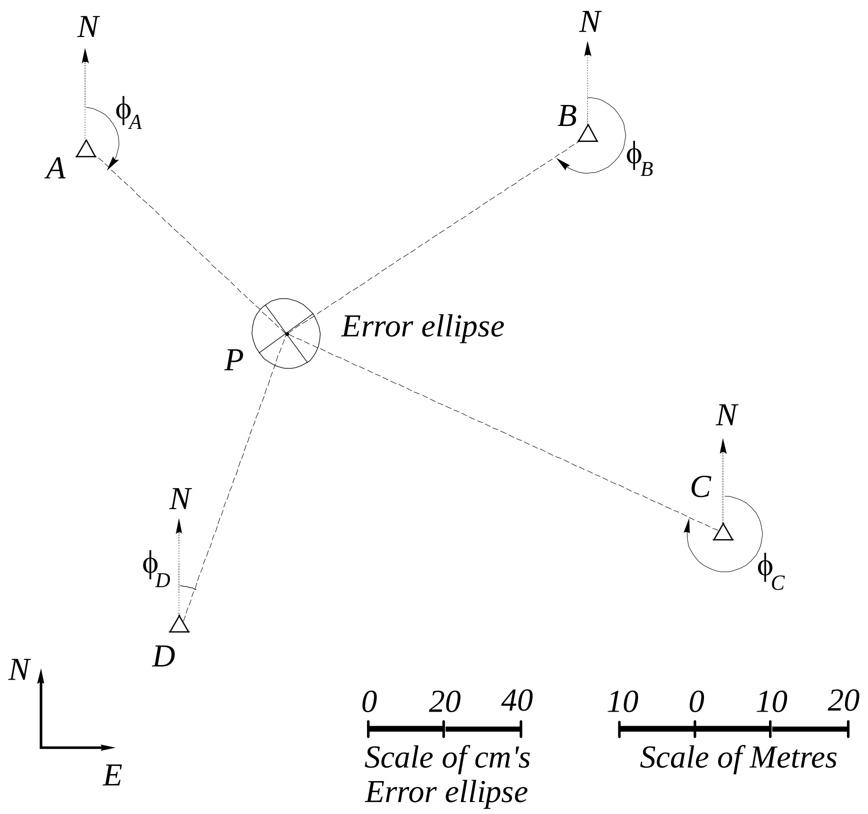

Figure 2. Athlete at *P* and error ellipse indicating precision of fix

ANOTHER EXAMPLE

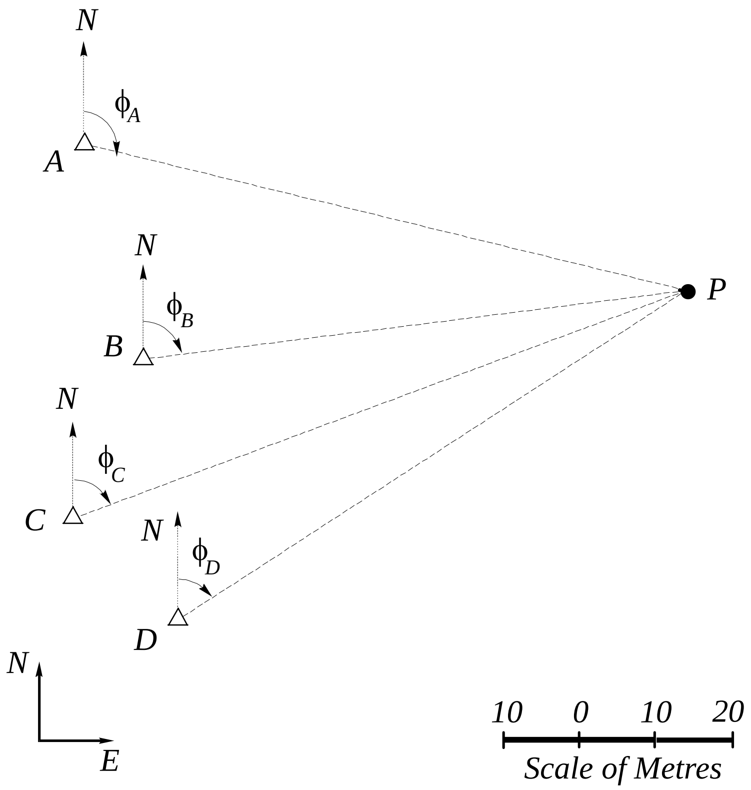

Figure 3. Athlete at *P* and bearings from antennae locations *A*, *B*, *C* and *D*.

To demonstrate the properties of the error ellipse consider Figure 3 which shows an athlete at *P* and the four antennae locations *A*, *B*, *C* and *D*. The antennae are all to the left of the athlete and not as well spread as in the first example. The data for the observed bearings and the antennae locations are

Point	East	North
A	6.00	79.06
B	16.00	50.50
C	4.50	29.50
D	18.36	16.15

Point	Bearing	Estimated st.dev. s_ϕ
ϕ_A	$103^\circ\,30'$	$0^\circ 10'$
ϕ_B	$82^\circ\,30'$	$0^\circ 10'$
ϕ_C	$69^\circ\,40'$	$0^\circ 10'$
ϕ_C	$57^\circ\,00'$	$0^\circ 10'$

A solution as per the previous example gives the parameters of the error ellipse as

$$a = 34.95 \text{ cm}$$

$$b = 24.74 \text{ cm}$$

$$\theta = 11.7 \text{ degrees}$$

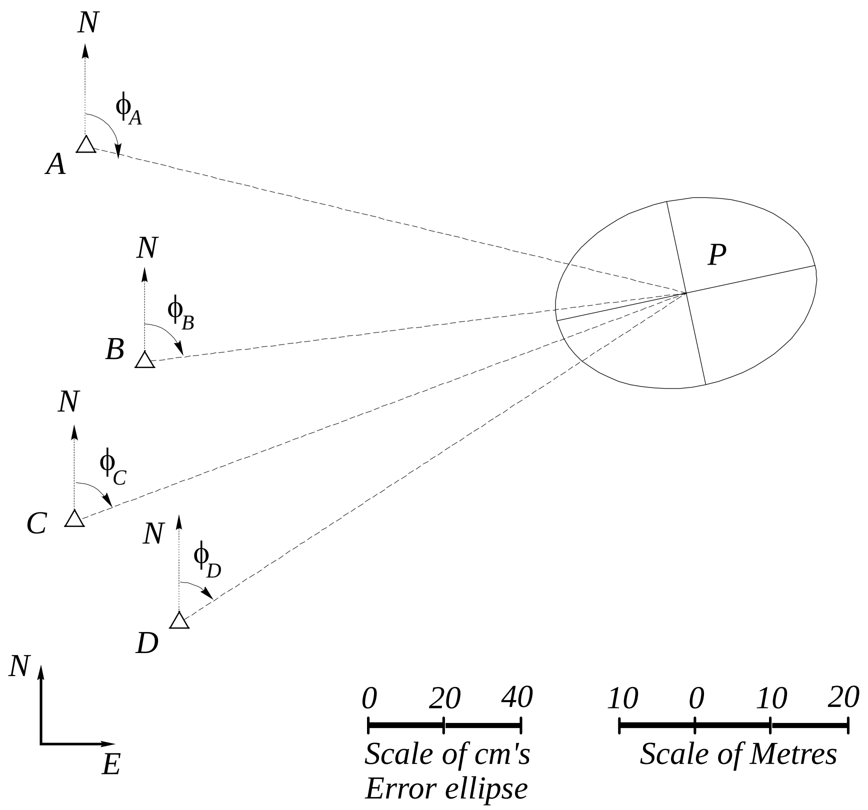

Figure 4. Athlete at *P* and error ellipse indicating precision of fix.

The size, shape and orientation of the error ellipse is a visual indication of the precision of the athlete's computed position. Comparing Figures 2 and 4 shows the relative precisions of the two solutions; the second example much worse than the first.

PLAYER ON A SOCCER FIELD

Suppose there are four radio direction antennae located near the corners of a soccer field 110 metres by 73 metres. The antennae are located at offsets of 10 metres from the corners of the field. The player, with a radio transmitter on their body, moves about the field and their position and the precision of the position fix can be determined from the four bearings using least squares. The parameters of an error ellipse can be computed and the size, shape and orientation of the ellipse indicate the precision of location of the athlete. For example, in Figure 5, locations at the corners are less precise than locations near the centre. Near circular ellipses indicate good precision of location and narrow elongated ellipses indicate poor precision.

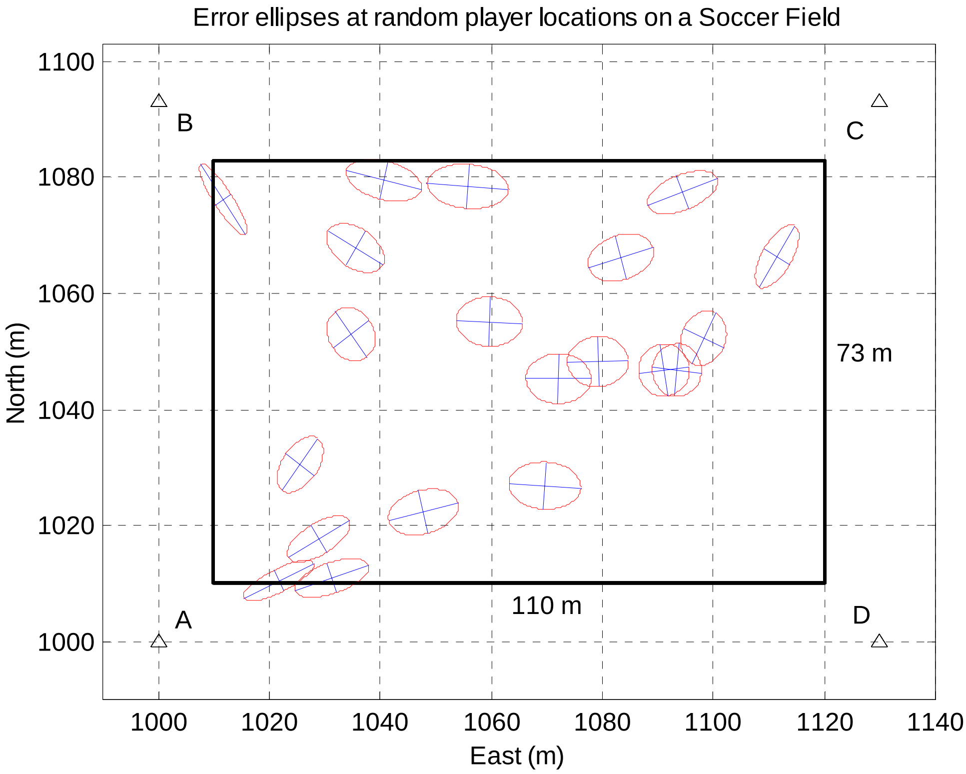

Figure 5. Error ellipses at random locations of a player on a soccer field. Standard deviation of bearings 1°. Antennae at *A*,*B*,*C* and *D*..

For a particular arrangement of antennae, error ellipses will have certain size, shape and orientation at different locations on the field. These ellipse characteristics may be used to compare different arrangements of antennae and different numbers of antennae. Two properties of error ellipses may be useful in this comparison.

The flattening f

$$f = \frac{a-b}{a} \tag{24}$$

a and b are the semi-major and semi-minor axes of the ellipse respectively. The flattening is the ratio of the difference in semi-axes lengths to the semi-major axis and varies between 0 (a circle) and 1 (a straight line). A near circular ellipse (indicating a good location fix) will have a small flattening and a narrow elongated ellipse (poor fix) will have a relatively large flattening.

The area A

$$A = \pi\, a\, b \tag{25}$$

a and b are the semi-major and semi-minor axes of the ellipse respectively. The area of an error ellipse can be used as a crude measure of relative precision. If the flattening of two ellipses were similar then the ellipse with the smaller area would indicate a better precision of position fix.

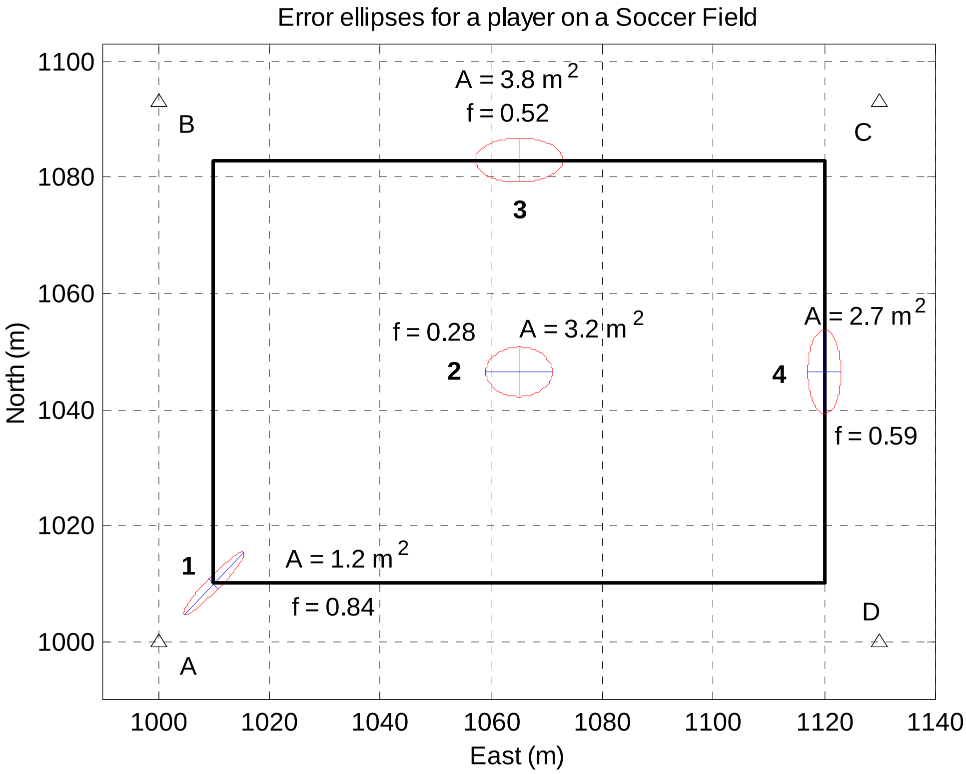

Figure 6. Area and flattening of error ellipses at four locations on a soccer field. Standard deviation of bearings 1°. Antennae at four corners *A*,*B*,*C* and *D.*

Figure 6 shows error ellipses for four location of a player, (1) at the south-west corner, (2) at the centre of the ground, (3) at the mid-point of the northern boundary and (4) at the mid-point of the eastern boundary. The flattening *f* and area *A* of the ellipses are

Ellipse	*f*	*A*
1	0.84	1.2 m^2
2	0.28	3.2 m^2
3	0.52	3.8 m^2
4	0.59	2.7 m^2

Table 1. Flattening *f* and area *A* of the error ellipses in Figure 6

The relatively large flattening for the ellipse at (1) indicates a poor position fix at the corners of the ground, for this configuration of antennae. The relatively small flattening for the ellipse at (2) indicates a good position fix at the centre of the ground. For the ellipses at (3) and (4), which have similar flattening we might conclude that the position fix at (4) is marginally better than that at (3) based on the areas of the ellipses.

An error ellipse can be computed for any position on the soccer field given the locations of the antennae and a grid of positions can be used to construct contour plots of error ellipse flattening. Figure 7 shows a contour plot of flattening derived from ellipse parameters computed at 1 metre intervals over the soccer field (110 × 73 = 8030 points).

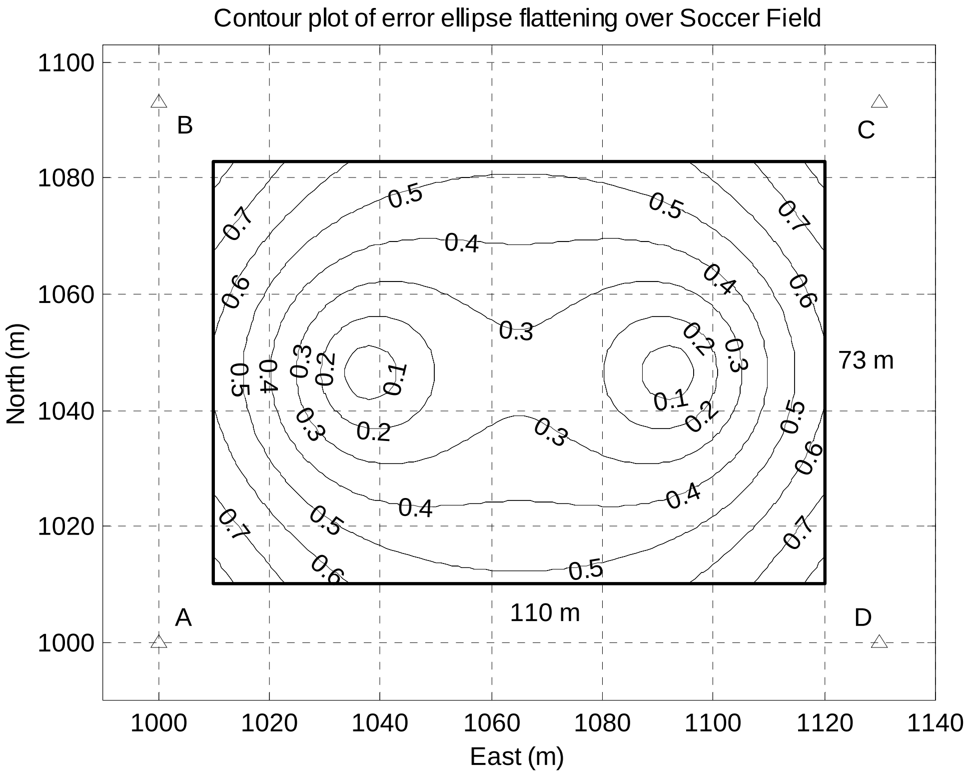

Figure 7. Contour plot of error ellipse flattening for computed positions over a soccer field. Standard deviation of bearings 1°. Antennae at four corners *A*,*B*,*C* and *D*.

The average area of the error ellipses is 2.65 m^2 and the average flattening is 0.43.

A different arrangement of the four antennae is shown in Figure 8 with error ellipses at random locations across the soccer field. In Figure 8 the four antennae are located at the mid-points of a rectangle whose sides are parallel to the field and 10 metres from the field boundaries.

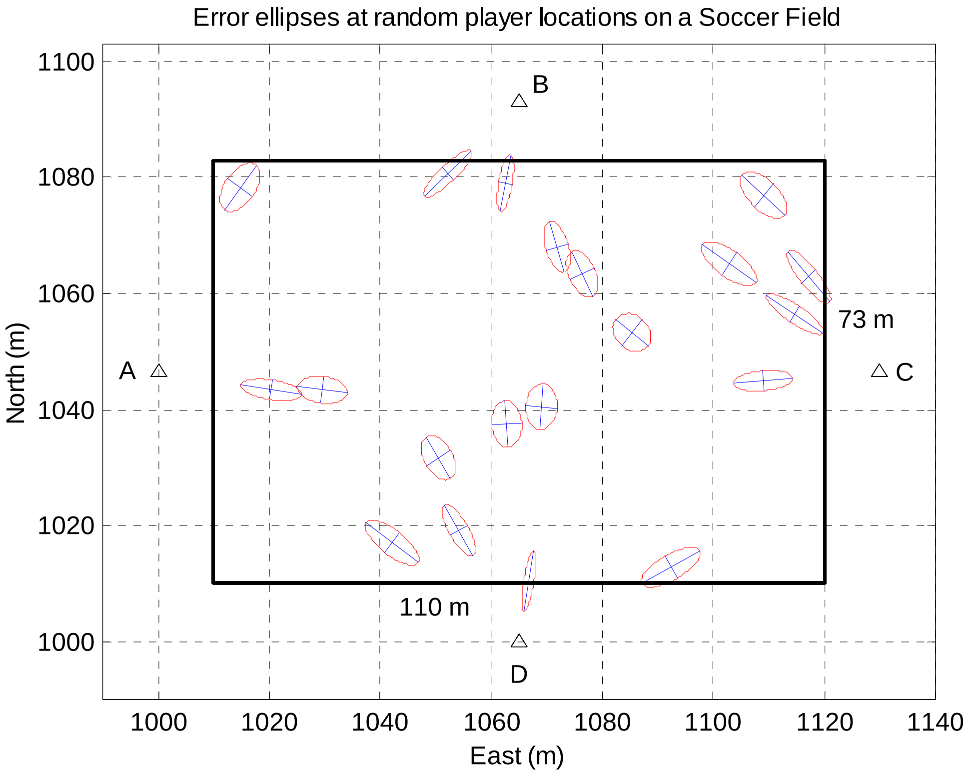

Figure 8. Error ellipses at random locations of a player on a soccer field. Standard deviation of bearings 1°. Antennae on four sides.

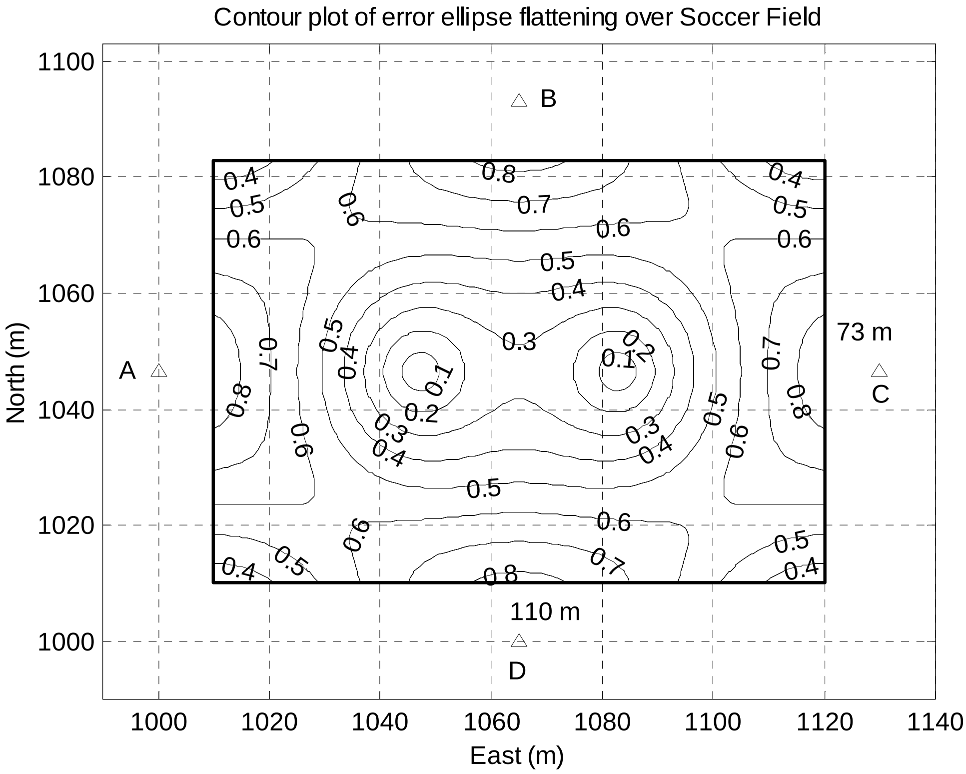

Figure 9. Contour plot of error ellipse flattening for computed positions over a soccer field. Standard deviation of bearings 1°. Antennae at four sides *A*,*B*,*C* and *D*.

The average area of the error ellipses is 1.41 m^2 and the average flattening is 0.53.
Another arrangement of eight antennae is shown in Figure 10 where the antennae are located equidistantly along the sides of a rectangle whose sides are parallel to the field and 10 metres from the field boundaries.

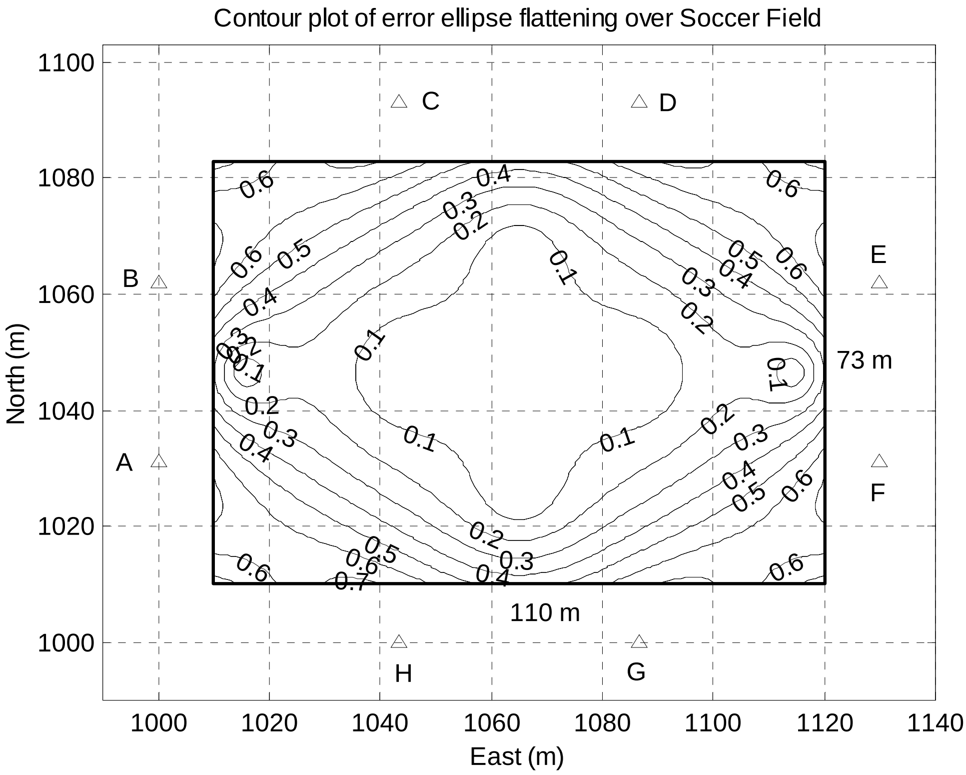

Figure 10. Contour plot of error ellipse flattening for computed positions over a soccer field. Standard deviation of bearings 1°. Antennae in pairs along four sides.

The average area of the error ellipses is 0.59 m^2 and the average flattening is 0.32.

Another arrangement of twelve antennae is shown in Figure 11. Eight antennae are located equidistantly along the sides of a rectangle whose sides are parallel to the field and 10 metres from the field boundaries and four antennae are located at the corners.

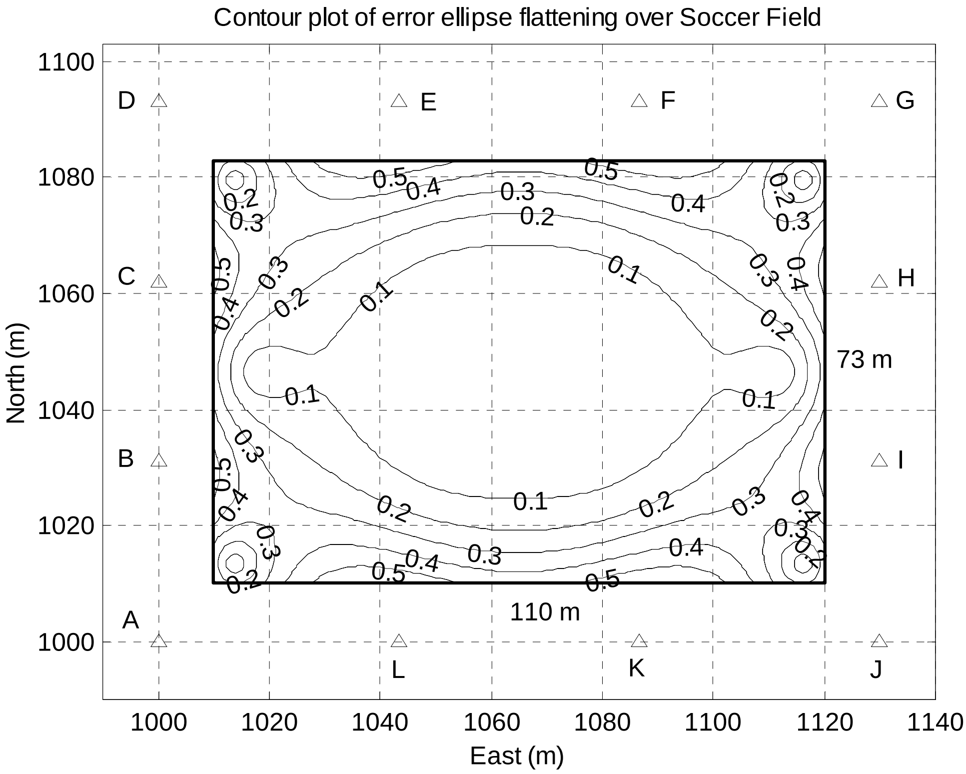

Figure 11. Contour plot of error ellipse flattening for computed positions over a soccer field. Standard deviation of bearings 1°. Twelve antennae in pairs along four sides and at the corners.

The average area of the error ellipses is 0.43 m^2 and the average flattening is 0.21.

This arrangement (12 antennae) has the smallest average area and average flattening of the error ellipses over the soccer field. A tabulation of antennae arrangements is shown in Table 2.

Number of antennae	Arrangement of antennae	Contour plot	Average area of error ellipses (m^2)	Average flattening of error ellipses
4	corners	Figure 7	2.65	0.43
4	sides	Figure 9	1.41	0.53
8	sides	Figure 10	0.59	0.32
12	sides + corners	Figure 11	0.43	0.21

Table 2. Antennae arrangements and average areas and flattening of error ellipses.

PRECISION OF DIRECTIONAL ANTENNAE BEARINGS FOR DESIRED ATHLETE PRECISION FIX

Suppose that it is desired to fix the position of the athlete on the soccer field to ±1 m in horizontal position. What precision is required for the directional antennae bearings to the player?

This question can only be answered if the number and arrangement of antennae is known. From the previous contour plots of error ellipse flattening (see Table 2) it appears that 12 antennae (8 along sides and 4 at the corners of a bounding rectangle) gives the smallest average error ellipse flattening. This could be regarded as the beast of the four antennae arrangements investigated. Each of these four arrangements assumed a precision (standard deviation) of measured bearings of 1°. If the standard deviation was increased to 5° the average area of the error ellipses (computed at 1 metre intervals over the field, 8030 locations) increases to 10.83 m2 with the average flattening remaining unchanged at 0.21. Since the area if the ellipse is $\pi a b$, then $a b = 3.45$. If the ellipses were regarded as approximately circular then the radius of these circles would be $r = 1.86\,\mathrm{m}$ which may be regarded as a crude estimate of the precision of the position fix if the precision of the antennae bearings were 5°. For the 8030 computed athlete positions, the mean of the estimated standard deviations of east coordinates was 1.92 m and the mean of the estimated standard deviations of the north coordinates was 1.83 m. This indicates that our crude position estimate ±1.86 m derived from the average area of the ellipses is reasonable. Table 3 shows values for the 12 antennae arrangement.

St. Dev. of bearing	Average area of error ellipse (m^2)	Circular position precision *r*	Average St. Dev of East coord.	Average St. Dev. of North coord.
1°	0.43	0.37	0.38	0.37
2°	1.73	0.74	0.77	0.73
3°	3.90	1.11	1.15	1.10
4°	6.93	1.49	1.54	1.47
5°	10.83	1.86	1.92	1.83

Table 3. Precision of position fix for 12 antennae arrangement.

From Table 3 we may conclude that for the 12 antennae arrangement, the bearings to the athlete would have to have a precision of 3° to achieve an average precision of ±1 m in the athlete's position.

REFERENCES

Mikhail, E.M., 1976. *Observations and Least Squares,* IEP—A Dun-Donnelley, New York.